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THE EFFECT OF DAMPING ON THE STABILITY PROPERTIES OF EQUILIBRIA OF NON-AUTONOMOUS SYSTEMS†

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A review of recent results on the different properties of stability and stability with respect to a part of variables for a damped oscillator is presented. Asymptotic stability with respect to velocities is guaranteed for the equilibrium of Lagrange systems acted upon by friction with unlimited damping factors. Instances of the scalar equations

$$\ddot{x}+h(t)\dot{x}+x=0 \text{ and } \ddot{x}+h(t,x,\dot{x})\dot{x}+f(x)=0$$

are considered in the case of "large" damping, "small" damping and in the "general" case. The effect of "intermittent" friction is investigated.

1. DAMPED OSCILLATORS

Consider a holonomic scleronomous mechanical system with R degrees of freedom, subject to potential, dissipative and gyroscopic forces. As is well known, the motions are described by Lagrange's equations

$$\frac{\partial}{dt}\frac{\partial T}{\partial \dot{q}}-\frac{\partial T}{\partial q}=-\frac{\partial \Pi}{\partial q}+Q \tag{1.1}$$

where we have used the following notation: the column vector $q, \dot{q} \in \mathbb{R}^r$ consist of generalized coordinates and velocities respectively (q^T is the row vector obtained by transformation of $q \in \mathbb{R}^r$), the potential energy II: $q \mapsto \Pi(q) \in \mathbb{R}$ is continuously differentiable, where $\Pi(0) = 0$, the kinetic energy has the form $T = T(q, \dot{q}) = (½)\dot{q}^T A(q)\dot{q}^T$, where the symmetrical matrix function A: $q \mapsto A(q) \in \mathbb{R}^{r\times r}$ is continuously differentiable, the continuous function Q: $(q, \dot{q}) \to Q(q, \dot{q}) \in \mathbb{R}^r$ is the resultant of non-potential and dissipative forces with complete dissipation, i.e. a function $c \in \mathcal{K}$ exists (where $\mathcal{K}$ is a class of strictly increasing continuous functions w: $\mathbb{R}_+ := [0, \infty] \to \mathbb{R}_+$ with $w(0) = 0$), such that $Q^T(q, \dot{q})\dot{q} \leq -c(|\dot{q}|)$ for all $q, \dot{q} \in \mathbb{R}^r$. We will assume that $q = \dot{q} = 0$ is the equilibrium of system (1.1). We will consider the stability properties of this equilibrium.

A special case of system (1.1) was considered in [1], which describes the motion of a point mass of unit mass in $\mathbb{R}^3$

$$\ddot{x}=\frac{\partial U}{\partial x}-f(v)\frac{\dot{x}}{v},\ \ddot{y}=\frac{\partial U}{\partial y}-f(v)\frac{\dot{y}}{v},\ \ddot{z}=\frac{\partial U}{\partial z}-f(v)\frac{\dot{z}}{v} \tag{1.2}$$

$$v:=((\dot{x})^2+(\dot{y})^2+(\dot{z})^2)^{½},\ f(v)>0 \text{ for } v>0,\ f(0)=0$$

where $U = U(x, y, z)$ is the potential function, v is the magnitude of the velocity, $f = f(v)$ is the damping factor, and the following two theorems were proved.

Theorem A [1]. If the potential function U has an isolated maximum at the point

$$x=y=z=0 \tag{1.3}$$

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then the equilibrium

$$x = y = z = \dot{x} = \dot{y} = \dot{z} = 0 \tag{1.4}$$

is stable.

Theorem B [1]. If the potential function U has an isolated minimum at the point (1.3) and the ratio $f(v)/v$ has an upper limit near to $v = 0$, equilibrium (1.4) is unstable.

Fejér writes [1]: "If the friction is large in the sense that the order of $f(v)$ is v^α $(0 < \alpha < 1)$ near to $v = 0$, then the method of my proof does not work. I have to leave open the question whether friction is able to cease the instability in this case".

These theorems were only the first stage in solving the problem of generalizing the Lagrange–Dirichlet theorem and its inverse for dissipative systems (see [2–10]). Thus, for the conditions of theorem A we would expect more, namely: asymptotic stability of the equilibrium. Fejér only noted that "velocity v becomes arbitrarily small infinitely many times along the motions"; in other words, $\liminf v(t) = 0$ as $t \to \infty$. Rumyantsev [4] proved that, more accurately, $\lim v(t) = 0$ when $t \to \infty$ for the more general system (1.1).

Theorem C [4]. Suppose $\Pi(q) \equiv 0$ in system (1.1). Then the equilibrium $q = \dot{q} = 0$ is asymptotically stable with respect to velocities.

We can also expect the same property when potential forces act, as in system (1.2). We will investigate the problem for system (1.1) in the case when the dissipative forces may also depend explicitly on the time t.

Theorem 1.1 [11]. If the potential energy Π in system (1.1) has a strict minimum at $q = 0$ and the following relations hold

$$Q = Q(q, \dot{q}, t) = B(q, t)\dot{q}, \quad \dot{q}^T B(q, t)\dot{q} \geqslant \dot{q}^T B(q)\dot{q}$$

with a positive-definite matrix function $B(q)$ in a neighbourhood of the equilibrium $q = \dot{q} = 0$ for all $t \geqslant 0$, this equilibrium is stable, asymptotically stable with respect to the velocities and, moreover, the system "asymptotically stops" [8].

The latter means that for each motion, which begins from a small neighbourhood of the equilibrium, the following relations hold

$$\lim \dot{q}(t) = 0, \quad \lim q(t) = q_* \in \mathbb{R}^r \quad \text{as} \quad t \to \infty$$

The following question arises: can the system stop asymptotically at a point $q_* \neq 0$ which is not an equilibrium position? This problem is considered in the following section. Note that the question which Fejér left open is not such a great one. It follows from Salvadori's theorem [10, Theorem 5.2 in Chapter III], that theorem B is true for an arbitrary continuous function f: $\mathbb{R}_+ \to \mathbb{R}_+$. It should be noted that Salvadori's theorem is based on the Barbashin–Krasovskii theorem on instability [12].

2. LARGE DAMPING

The answer to the question posed at the end of the previous section, is in the affirmative. Consider the linear second-order differential equation

$$\ddot{x} + (2 + e^t)\dot{x} + x = 0$$

The function $x(t) = c(1 + e^{-1})$ is the solution for each $c \in \mathbb{R}$ and $\lim x(t) = c$ as $t \to \infty$. This is the phenomenon of *overdamping*: the damping factor increases so rapidly that the point cannot return to equilibrium since the friction may compensate the large potential force, despite the fact that the velocity approaches zero. We ask, is this phenomenon maintained when the damping factor approaches infinity as $t \to \infty$? The answer is in the negative. It can be shown that the equation

$$\ddot{x} + t\dot{x} + x = 0$$

has the general solution

$$x(t) = \exp\left(-\frac{t^2}{2}\right)\left\{c_1 + c_2 \int_0^t \exp\frac{s^2}{2}\,ds\right\}$$

and, consequently, all the solutions approach zero as $t \to \infty$. Hence, the following problem arises: what conditions must be imposed on the matrix $B(q, t)$ of the damping factors so as to guarantee that $q_* = 0$ for all solutions or, in other words, that the system stops asymptotically only in an equilibrium position?

In order to investigate this problem we must begin with the "kinetic energy rule":

$$\dot{q}^T(t)A(q(t))\dot{q}(t) + \Pi(q(t)) = -\int_{t_0}^{t} \dot{q}^T(s)B(q(s), s)\dot{q}(s)ds + \text{const}$$

and attempt to make the right-hand side of this equation approach zero as $t \to \infty$. The difficulty lies in the need to estimate the integral on the right-hand side, knowing only the function $(q, t) \mapsto B(q, t)$ for unknown motions $t \mapsto (q(t), \dot{q}(t))$.

We will consider the simplest case of a linear system with one degree of freedom

$$\ddot{x} + h(t)\dot{x} + x = 0, \quad h(t) \geqslant 0 \tag{2.1}$$

In this case, the asymptotic stability of the equilibrium

$$x = \dot{x} = 0 \tag{2.2}$$

of system (2.1) is equivalent to the conditions

$$\lim x(t) = \lim \dot{x}(t) = 0 \quad \text{as} \quad t \to \infty \quad \text{for all solutions} \tag{2.3}$$

We know from stability theory that if $h(t) \equiv h_0 = \text{const}$, $h_0 > 0$, the zero solution of system (2.1) is asymptotically (exponentially) stable. This fact can be generalized as follows:

Theorem D [13]. Assume that constants $\underline{h}, \overline{h}$ exist such that

$$0 < \underline{h} \leqslant h(t) \leqslant \overline{h} < \infty \quad \text{for all} \quad t \geqslant 0 \tag{2.4}$$

Conditions (2.3) then hold.

In view of condition (2.4) we can distinguish three cases:

1) large damping: $0 < \underline{h} \leqslant h(t)$ $(t \geqslant 0)$,
2) small damping: $0 \leqslant h(t) \leqslant \overline{h} < \infty$ $(t \geqslant 0)$,
3) the general case: $0 \leqslant h(t) < \infty$ $(t \geqslant 0)$.

In the case of large damping we need to eliminate overdamping. We will give the necessary and sufficient conditions for relation (2.3) to be satisfied using the integral function

$$H(t) = \int_0^t h(s)ds \ (t \geqslant 0)$$

Theorem 2.1 [14]. Assume that $h(t) \geqslant \underline{h} > 0$ for all $t \geqslant 0$ and c is a fixed positive number. Then equilibrium (2.2) of system (2.1) is asymptotically stable if and only if

$$\Sigma \equiv \sum_{n=1}^{\infty} [H^{-1}(nc) - H^{-1}((n-1)c)]^2 = \infty \tag{2.5}$$

where H^{-1} is the inverse of H.

The advantage of condition (2.5) is that it is not only necessary and sufficient but also that it can easily be checked. In fact,

$$\Sigma \equiv \sum_{n=1}^{\infty} [(2n)^{1/2} - (2(n-1))^{1/2}]^2 \geqslant \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n} = \infty, \quad \text{if} \quad h(t) = t$$

$$\Sigma \equiv \sum_{n=1}^{\infty} [(3n)^{1/3} - (3(n-1))^{1/3}]^2 \leqslant 3\sum_{n=1}^{\infty} \frac{1}{n^{4/3}} < \infty, \quad \text{if} \quad h(t) = t^2$$

i.e. the equilibrium is asymptotically stable for $h(t) = t$ and is not asymptotically stable for $h(t) = t^2$.

From *Theorem* 2.1 we can conclude that the condition

$$\int_0^{\infty} \frac{dt}{h} = \infty \tag{2.6}$$

is necessary for asymptotic stability. It was shown above that this condition is not sufficient. However, using Theorem 2.1, the sufficient conditions can be obtained in terms of the function h.

Corollary [14]. Assume that $h(t) > 0$ for all $t \geqslant 0$ and that condition (2.6) is satisfied. If the function h is differentiable and the quantity $(1/h)'$ has an upper or lower limit, equilibrium (2.2) of Eq. (2.1) is asymptotically stable.

All the previous conditions imposed on the increase in h and which guarantee asymptotic stability can easily be derived from Theorem 2.1 (see [14]).

Note that Theorem 2.1 only solves the problem for linear scalar equation (2.1). It is important to obtain the condition corresponding to condition (2.5) for linear systems and for Lagrange's equations (1.1).

3. SMALL DAMPING

The case of small damping is more complex. Here there are no more necessary and sufficient conditions, overdamping does not occur, and non-oscillating solutions approach zero, so that only oscillating solutions are of interest. It follows from the kinetic energy rule that the condition $H(\infty) = \infty$ is necessary to satisfy condition (2.3). It was proved in [15] that this condition is also sufficient for the monotonic function h. However, in practice damping is often intermittent and hence the non-monotonic case is of particular interest. The following theorem gives a solution of the problem for the coefficient h equal "on average" to a positive constant.

Theorem E [16]. We will assume that $0 \leqslant h(t) \leqslant \bar{h} = \text{const}$ for all $t \geqslant 0$ and there are constants $B > 0$, T such that

$$H(t) \geqslant Bt, \; t \geqslant T$$

The zero solution of system (2.1) is then asymptotically stable.

We ask, can this condition be improved? The following theorem gives a negative answer to this question and leads to a result that cannot be improved for power functions of h.

Theorem 3.1 [17]. 1. If

$$\limsup_{t\to\infty} t^{-2/3} H(t) > 0 \tag{3.1}$$

equilibrium (2.2) of system (2.1) is asymptotically stable.

2. For each $\varepsilon > 0$ a function h exists such that

$$\liminf_{t\to\infty} t^{-2/3+\varepsilon} H(t) > 0$$

but nevertheless the zero solution of system (2.1) is not asymptotically stable.

The kinetic energy rule shows that the action of friction can be measured by the integral of the damping factor h. From this point of view condition (3.1) is more preferable than $h(t) \geqslant h$ in Theorem D of Levin–Nohel, but it requires further improvement for the following reason. We will assume that the coefficient h satisfies condition (3.1), so that condition (2.3) is satisfied. We modify h in accordance with the following definition:

$$\tilde{h}(t) = \begin{cases} h(t), & 0 \leqslant t < \tilde{t} \\ 0, & \tilde{t} \leqslant t \leqslant \tilde{t} + 2k\pi \\ h(t - 2k\pi), & t > \tilde{t} + 2k\pi \end{cases}$$

where the quantity $\tilde{t} > 0$ is fixed arbitrarily and k is an arbitrary natural number. Since system (2.1) describes an harmonic oscillator with 2π-periodic motions in the interval $[\tilde{t}, \tilde{t} + 2k\pi]$, equilibrium (2.2) remains asymptotically stable for arbitrary $\tilde{t}, k$. Moreover, this modification can also be realised for an arbitrary set of values of the time $\tilde{t}'$. This means that we can attain the fact that the equilibrium has remained asymptotically stable but condition (3.1) no longer holds. In other words, condition (3.1) is far from being the necessary condition but must be improved for intermittent damping.

Definition. Damping is said to be *intermittent* if there is a sequence $\{I_k\}_{k=1}^{\infty}$ of disjoint intervals such that there is no information on the friction outside these intervals.

The first theorem on intermittent damping was a generalization of Theorem D of Levin–Nohel.

Theorem F [18]. We will assume that $0 < h \leqslant h(t) \leqslant \bar{h} < \infty$ when $t \in \cup_{k=1}^{\infty} I_k$. If

$$\sum |I_k|^3 = \infty \tag{3.2}$$

where $|I_k|$ is the length of the interval I_k (everywhere henceforth summation is carried out from $k = 1$ to infinity), then equilibrium (2.2) of system (2.1) is asymptotically stable. Moreover, the exponent three in condition (3.2) is unimprovable.

In this theorem the condition $h(t) \geqslant h > 0$ does not agree with the principle according to which the damping effect can be measured by the integral of the function $h(t)$. For this reason Theorem D can be deduced from Theorem F, while Theorem E can not. We will now formulate a theorem which implies both Theorems D and E.

Theorem 3.2 [17]. If a sequence $\{I_k\}_{k=1}^{\infty}$ of disjoint intervals exists and also there is a constant $\bar{h} < \infty$ such that

$$0 \leqslant h(t) \leqslant \bar{h} < \infty \;\; t \geqslant 0, \;\; \sum J_k^{(3)} / (1 + |I_k|^2) = \infty \tag{3.3}$$

$$\left(J_k^{(3)} = \left(\int_{I_k} h(t)dt \right)^3 \right)$$

then equilibrium (2.2) of Eq. (2.1) is asymptotically stable, i.e. relation (3.1) is satisfied.

Corollary. If condition (3.3) holds, the quantity $\{I_k\}_{k=1}^{\infty}$ is bounded and $\Sigma J_k^3 = \infty$, relations (2.3) are satisfied. The exponent three in this assertion is unimprovable.

Note that Assertion 1° in Theorem 3.1 can be deduced from Theorem 3.2. To do this it is sufficient to prove that relation (3.1) implies the existence of the series $\{I_k\}_{k=1}^{\infty}$ and a quantity $\gamma > 0$ such that

$$J_k^3 / (1 + |I_k|^2)^3 \geqslant \gamma, \;\; k = 1, 2, \ldots \tag{3.4}$$

since condition (3.3) is then obviously satisfied.

If condition (3.1) is satisfied, *a* $\delta > 0$ exists such that for each T we can obtain a $t_* > T$ for which

$$t_*^{-2/3} H(t_*) \geqslant \delta$$

Suppose $\gamma = (\delta/4)^3$ and let us choose a number a. We seek $b \geqslant a + 1$ such that relation (3.4) is satisfied with this value of γ and for this interval $I_k = (a, b)$. If b is so large that

$$b^{-2/3} H(a) < \delta/2, \;\; b^{-2/3} H(b) \geqslant \delta$$

then

$$(1 + (b - a)^2)^{-1/3} (H(b) - H(a)) \geqslant \delta/2 - \delta/4 = \delta/4 = \gamma^{1/3}$$

which indicates that inequality (3.4) is satisfied.

4. THE GENERAL CASE

In order to proceed to an investigation of Lagrange's equations (1.1), we will consider the non-linear equation

$$\ddot{x} + h(t,x,\dot{x})\dot{x} + f(x) = 0,\quad h: \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+,\quad f: \mathbb{R} \to \mathbb{R} \tag{4.1}$$

where the function h is specified, the function f is continuous and $f(x)x > 0$, if $x \neq 0$. We will assume that the following inequalities are satisfied for the damping factor h

$$a(t) \leqslant h(t,x,y) \leqslant b(t) \qquad (x^2 + y^2 \leqslant \Delta,\ t \geqslant 0)$$

with certain continuous functions a, b: $\mathbb{R}_+ \to \mathbb{R}_+$.

The following generalization of Theorem 3.2 holds.

Theorem 4.1. If a sequence $\{I_k\}_{k=1}^{\infty}$ of disjoint intervals exists, such that

$$\sum \left(\int_{I_k} a\,dt \right)^3 (1 + |I_k|^2)^{-1} \left(\sup_{I_k} b \right)^{-2} \left(1 + \min\left\{ \sup_{I_k} a;\ \int_{I_k} a\,dt \right\} \right)^{-2} = \infty \tag{4.2}$$

then equilibrium (2.2) of system (4.1) is asymptotically stable.

The question arises of how sharp is condition (4.2). As already stated above, necessary and sufficient conditions do not exist even for the case of small damping, apart from one special case of linear equation (2.1) when the coefficient h is a step function:

$$h(t) = \begin{cases} h_k > 0, & t \in I_k,\ k = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

where $\{h_k\}_{k=1}^{\infty}$ is a specified sequence of real numbers and $I_k = [\alpha_k, \beta_k]$, $\alpha_k < \beta_k < \alpha_{k+1}$ $(k = 1, 2, \ldots)$ is a specified sequence of intervals. Equation (2.1) with this coefficient is integrable: it can be solved on the intervals $[\alpha_k, \beta_k]$ and $[\beta_k, \alpha_{k+1}]$, and the portions of the solutions can be joined continuously. The following necessary and sufficient conditions were obtained by this method in [19].

Theorem G [19]. Assume that the sequences $\{|I_k|\}_{k=1}^{\infty}$ and $\{h_k|I_k|\}_{k=1}^{\infty}$ are bounded. Equilibrium (2.2) of Eq. (2.1) with coefficient (4.3) is asymptotically stable if and only if

$$\sum h_k |I_k|^3 = \infty \tag{4.4}$$

We will formulate a corollary of Theorem 4.1 for a step function of h.

Corollary. If

$$\frac{\sum h_k |I_k|^3}{(1 + |I_k|^2)(1 + \min\{h_k : h_k |I_k|\})^2} = \infty \tag{4.5}$$

then equilibrium (2.2) of the equation

$$\ddot{x} + h(t)\dot{x} + f(x) = 0$$

with coefficient (4.3) is asymptotically stable.

It can be seen that when the boundedness conditions are satisfied in Theorem G, condition (4.5) reduces to condition (4.4). Since condition (4.4) is necessary and sufficient, this means that condition (4.2) is close to unimprovable.

Omitting the lengthy proof, we note that the main idea is as follows. As already pointed out above, we need to estimate the integral

$$\int_0^{\infty} h(t)(\dot{x}(t))^2\,dt$$

without knowing the solutions. Instead of this integral we estimate

$$\int_0^\infty h(t)\xi^2(t)dt$$

where ξ is a typical element of the well-chosen functional space [20].

The general case of Lagrange systems (1.1) was considered previously in [21, 22].

From 1971 to 1974 I was a graduate student in the Department of Theoretical Mechanics at Moscow State University, where, under the supervision of V. V. Rumyantsev, I acquired basic knowledge in the field of theoretical mechanics. I regard the scientific school headed by Professor Rumyantsev as the best school in this field.

While I was a postgraduate student I studied dissipative effects in the case of non-stationary laws of friction, which are the basis of complex problems in mathematics and dynamics. I became familiar with these problems at Rumyantsev's school, but, as I found out later, they had already been investigated by prominent Hungarian mathematicians in the Nineteenth Century. The aim of the present paper was to give a brief review of recent results obtained on oscillators with damping, while indicating their starting point in the Hungarian and Russian literature.

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